

ON QUASICONFORMAL EQUIVALENCE BETWEEN CERTAIN INFINITELY OFTEN PUNCTURED PLANES

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ABSTRACT. A closed discrete subset $A \subset \mathbb{C}$ is called “tame” if $\mathbb{C} \setminus A$ is quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$. By giving several criteria for A to be tame, we shall show that $\mathbb{Z} + i\mathbb{Z}$ is not tame.

1. INTRODUCTION

Let R be a Riemann surface. The Teichmüller space $T(R)$ is a space which describes all quasiconformal deformations of R . It is well known that $T(R)$ becomes either a finite dimensional complex manifold or a non-separable infinite dimensional Banach analytic manifold. $T(R)$ becomes finite dimensional if and only if R is of finite type. Through the investigation of quasiconformal deformations of a certain infinite type Riemann surface, a certain characteristic subspace will be found, which is separable.

The universal Teichmüller space $T(\mathbb{D})$ simultaneously describes all quasiconformal deformations of all hyperbolic type Riemann surfaces. This arises from the fact that each covering $X \rightarrow Y$ induces an embedding of $T(Y)$ into $T(X)$. On the other hand, $\mathbb{C} \setminus \mathbb{Z}$ covers a certain n -punctured Riemann sphere for each $n \geq 3$. Namely $T(\mathbb{C} \setminus \mathbb{Z})$ simultaneously describes all quasiconformal deformations of Riemann surfaces of genus 0 with at least three punctures. Needless to say, the universal Teichmüller space $T(\mathbb{D})$ also describes them. However for the reasons mentioned below, the Teichmüller space $T(\mathbb{C} \setminus \mathbb{Z})$ is more suitable to describe them than $T(\mathbb{D})$.

For each positive integer n , let $R_n = (\mathbb{C} \setminus \mathbb{Z}) / \langle z + n \rangle$. R_n is an $(n+2)$ -punctured Riemann sphere, and the projection $p^n : \mathbb{C} \setminus \mathbb{Z} \rightarrow R_n$ induces the embedding $p_*^n : T(R_n) \hookrightarrow T(\mathbb{C} \setminus \mathbb{Z})$. The covering transformation group of p^n is the cyclic group $\langle z + n \rangle$, so that, quasiconformal deformations of R_n correspond to periodic quasiconformal deformations of $\mathbb{C} \setminus \mathbb{Z}$ with only a period $z + n$. Then it is shown from McMullen’s theorem in [5] that p_*^n is totally geodesic for Teichmüller metric. By contrast, the embedding of $T(R_n)$ into $T(\mathbb{D})$ is not totally geodesic (cf. The Kra–McMullen theorem [5]). Additionally, $\mathbb{C} \setminus \mathbb{Z}$ is considered to be one of the smallest Riemann surface which has the above properties, that is, there exists no Riemann surface except $\mathbb{C} \setminus \mathbb{Z}$ which is covered by $\mathbb{C} \setminus \mathbb{Z}$ and covers R_n for all n .

Thus, in this paper, we would like to investigate quasiconformal deformations of $\mathbb{C} \setminus \mathbb{Z}$. First, we shall try to find all Riemann surfaces which are quasiconformally

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equivalent to $\mathbb{C} \setminus \mathbb{Z}$.

If R is quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$, then R is conformally equivalent to $\mathbb{C} \setminus A$ by a certain closed discrete subset $A \subset \mathbb{C}$ (cf. The removable singularity theorem, see [7, Theorem 17.3.] and [2, p.14]).

Definition . A closed discrete subset $A \subset \mathbb{C}$ is called *tame* if $\mathbb{C} \setminus A$ is quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$.

We consider the following problem.

Problem . Let \mathcal{P} be the family of all closed discrete infinite subsets $A \subset \mathbb{C}$. Find all tame $A \in \mathcal{P}$.

For this Problem, we obtain the following results as partial solutions.

First, for $A \in \mathcal{P}$, $z \in \mathbb{C}$, and $r > 0$, we define quantities d and \tilde{d} by

$$\begin{aligned} d(z, r; A) &= \sup_{w \in \bar{D}_r(z)} \text{dist}(w, A), \quad \bar{D}_r(z) = \{w \in \mathbb{C} \mid |w - z| \leq r\}, \\ \tilde{d}(z, r; A) &= \sup_{w \in Q_r(z)} \text{dist}(w, A), \quad Q_r(z) = \{z + x + iy \mid -r \leq x, y \leq r\}. \end{aligned}$$

Theorem A. Let $A \in \mathcal{P}$. If there exists a closed discrete subset $B \subset \mathbb{R}$ such that $\mathbb{C} \setminus B$ is quasiconformally equivalent to $\mathbb{C} \setminus A$, then

$$\sup_{z \in \mathbb{C}, r > 0} \frac{r}{d(z, r; A)}, \quad \sup_{z \in \mathbb{C}, r > 0} \frac{r}{\tilde{d}(z, r; A)} < +\infty.$$

Corollary . $\mathbb{Z} + i\mathbb{Z}$ is not tame.

The complex plane \mathbb{C} and the unit disk \mathbb{D} are well known as a typical example of Riemann surfaces which are homeomorphic and are not quasiconformally equivalent. This Corollary also gives an example of such Riemann surfaces, but is essentially different from the case of \mathbb{C} and \mathbb{D} .

In addition, various corollaries are provided by Theorem A. We shall introduce them in Section 2.3. To prove Theorem A, the fact that a quasidisk becomes a uniform domain plays an important role. This fact was proved by V. Gol'dšteĭn and S. Vodop'janov [4] first. Later, F. W. Gehring and B. Osgood proved the more generalized form in [3] and [6].

Next, we obtain the following theorem by comparing the extremal distances of certain continua. Let $D(a, r) = \{z \in \mathbb{C} \mid |z - a| < r\}$ for $a \in \mathbb{C}$ and $r > 0$.

Theorem B. Let $A \in \mathcal{P}$. Assume there exists an automorphism of infinite order $h \in \text{Aut}(\mathbb{C} \setminus A)$ such that the quotient space $(\mathbb{C} \setminus A)/\langle h \rangle$ has infinitely many punctures. If A is tame, then for any $\varepsilon > 0$ and $d \in \mathbb{N}$, there exists $a \in A$ such that

$$\#D(a, \varepsilon) \cap A \geq d,$$

where for a finite set X , $\#X$ denotes the number of elements of X .

Recall that A is closed and discrete in \mathbb{C} . Certainly, there exist some $A \in \mathcal{P}$ satisfying the above assumptions which cannot be decided whether A is tame or not from Theorem B, but they have to be a very strange form (see Example 3.5). Therefore it is expected that the next proposition holds.

Conjecture I. *Let $A \in \mathcal{P}$. If there exists an automorphism of infinite order $h \in \text{Aut}(\mathbb{C} \setminus A)$ such that the quotient space $(\mathbb{C} \setminus A) / \langle h \rangle$ has infinitely many punctures, then A is not tame.*

We can now rephrase Conjecture I as follows.

Let $T_0 = \bigcup_{n \in \mathbb{N}} p_*^n(T(R_n))$, namely T_0 is a subspace of $T(\mathbb{C} \setminus \mathbb{Z})$ which simultaneously describes all quasiconformal deformations of all Riemann surfaces of finite type $(0, n)$ with $n \geq 3$. Further, let \bar{T} be the set of all $[S, f] \in T(\mathbb{C} \setminus \mathbb{Z})$, the Teichmüller equivalence class of the quasiconformal homeomorphism $f : R \rightarrow S$, such that there exists an automorphism of infinite order in $\text{Aut}(S)$. Then Conjecture I implies

Conjecture II.
$$\bar{T} = \bigcup_{[f] \in \text{Mod}(\mathbb{C} \setminus \mathbb{Z})} [f]_*(T_0).$$

Here, $\text{Mod}(\mathbb{C} \setminus \mathbb{Z})$ is the Teichmüller-Modular group of $\mathbb{C} \setminus \mathbb{Z}$. The subspace T_0 is not closed in $T(\mathbb{C} \setminus \mathbb{Z})$. However, T_0 is separable by the definition, further geodesically convex with respect to the Teichmüller metric of $T(\mathbb{C} \setminus \mathbb{Z})$ by McMullen's theorem. Remark that, Conjecture I and II are not necessary to prove these properties.

2. PROOF OF THEOREM A

2.1. Quasidisks and uniform domains. A domain $D \subset \hat{\mathbb{C}}$ is called a quasidisk if there exists a quasiconformal homeomorphism $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ such that $f(\mathbb{D}) = D$, where $\hat{\mathbb{C}}$ and \mathbb{D} denote the Riemann sphere and the unit disk, respectively. Next, for a constant $c \geq 1$, a domain $D \subset \mathbb{C}$ is called a c -uniform domain if arbitrary two points $z_1, z_2 \in D$ can be joined by a rectifiable curve $\gamma \subset D$ which satisfies

- (1) $\ell(\gamma) \leq c |z_1 - z_2|$,
- (2) for all $z \in \gamma$, $\min_{j=1,2} \ell(\gamma[z, z_j]) \leq c \text{dist}(z, \partial D)$.

Here $\gamma[z, z_j]$ denotes the subcurve of γ which joins z and z_j . Further D is called a uniform domain if D is a c -uniform domain for some constant $c \geq 1$.

There is a lot of characterizations of quasidisks. Many of those are collected in [2] by F. W. Gehring. Actually, the uniformity of domains is one of the characterizations of quasidisks: for a simply connected proper subdomain $D \subset \mathbb{C}$, D is a quasidisk if and only if D is a uniform domain (cf. [4], [3] and [6]).

2.2. Proof of Theorem A.

Theorem A. *Let $A \in \mathcal{P}$. If there exists a closed discrete subset $B \subset \mathbb{R}$ such that $\mathbb{C} \setminus B$ is quasiconformally equivalent to $\mathbb{C} \setminus A$, then*

$$\sup_{z \in \mathbb{C}, r > 0} \frac{r}{d(z, r; A)}, \quad \sup_{z \in \mathbb{C}, r > 0} \frac{r}{\tilde{d}(z, r; A)} < +\infty.$$

Proof. $\bar{D}_r(z) \subset Q_r(z) \subset \bar{D}_{\sqrt{2}r}(z)$ holds for all $z \in \mathbb{C}$ and $r > 0$, so that, $d(z, r; A) \leq \tilde{d}(z, r; A) \leq d(z, \sqrt{2}r; A)$. Thus

$$\frac{1}{\sqrt{2}} \sup_{z \in \mathbb{C}, r > 0} \frac{r}{\tilde{d}(z, r; A)} \leq \sup_{z \in \mathbb{C}, r > 0} \frac{r}{d(z, r; A)} \leq \sup_{z \in \mathbb{C}, r > 0} \frac{r}{\tilde{d}(z, r; A)}.$$

Therefore it suffices to prove that Theorem A holds for d .

Let $f : \mathbb{C} \setminus B \rightarrow \mathbb{C} \setminus A$ be a quasiconformal homeomorphism. It is known from the removable singularity theorem that f can be extended to the quasiconformal homeomorphism from \mathbb{C} to \mathbb{C} (cf. see [7, Theorem 17.3.] and [2, p.14]). Moreover ∞ is also a removable singularity for extended f . Consequently, f can be extended to the quasiconformal homeomorphism $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ (we denote the extended f by the same letter f). Then $D = f(\mathbb{H})$ becomes a quasidisk, where \mathbb{H} denotes the upper half plane. Since $f(\infty) = \infty$, $f(\mathbb{H}) \subset \mathbb{C}$. Therefore D becomes a c -uniform domain for some constant $c \geq 1$. In the following argument, remark that the restriction $f|_B : B \rightarrow A$ is bijective.

Let $z_1 \in D$ and $r > 0$. Since D is not bounded, we can choose $z_2 \in D$ such that

$$|z_1 - z_2| = \frac{2r}{c}.$$

Then from the uniformity of D , there exists a rectifiable curve $\gamma \subset D$ which joins z_1 and z_2 with the properties that

- (1) $\ell(\gamma) \leq c |z_1 - z_2| = 2r$,
- (2) for all $z \in \gamma$, $\min_{j=1,2} \ell(\gamma[z, z_j]) \leq c \operatorname{dist}(z, \partial D)$.

By using the intermediate value theorem for $\gamma[z, z_1]$, it turns out there exists $z_0 \in \gamma$ such that $\ell(\gamma[z_0, z_1]) = \ell(\gamma)/2$. Then

$$\min_{j=1,2} \ell(\gamma[z_0, z_j]) = \ell(\gamma[z_0, z_1]) = \ell(\gamma[z_0, z_2]).$$

Therefore

$$2r \geq \ell(\gamma) = \ell(\gamma[z_0, z_1]) + \ell(\gamma[z_0, z_2]) = 2\ell(\gamma[z_0, z_1]) \geq 2|z_0 - z_1|.$$

Namely $z_0 \in \bar{D}_r(z_1)$. From the above,

$$\begin{aligned} \frac{2r}{c} = |z_1 - z_2| &\leq \ell(\gamma) \\ &= 2 \min_{j=1,2} \ell(\gamma[z_0, z_j]) \\ &\leq 2c \operatorname{dist}(z_0, \partial D) \leq 2c \operatorname{dist}(z_0, A) \leq 2c d(z_1, r; A). \end{aligned}$$

Thus we obtain $\frac{r}{d(z_1, r; A)} \leq c^2$. Recall that $z_1 \in D$ and $r > 0$ are arbitrary.

Moreover we can apply the above argument to $z_1 \in \partial D \setminus \{\infty\}$ and $r > 0$, because of Väisälä's theorem [8, Theorem 2.11.].

Finally, in the case of $z_1 \in \mathbb{C} \setminus \bar{D}$ and $r > 0$, we apply the same argument to the image of the lower half plane under f . Then it turns out there exists a constant $c' \geq 1$ which does not depend on z_1 and r , and satisfies

$$\frac{r}{d(z_1, r; A)} \leq c'^2.$$

Hence we obtain the claim. □

2.3. Some corollaries of Theorem A. In this section, we shall show some corollaries from Theorem A.

Corollary 2.1. *For arbitrary $s > 0$, the discrete set $A_s = \mathbb{Z} + i\{\pm n^s \mid n = 0, 1, 2, \dots\}$ is not tame. In particular $\mathbb{Z} + i\mathbb{Z}$ is not tame.*

Proof. If $s \leq 1$, it follows that $\tilde{d}(0, r; A_s) = \sqrt{2}/2$ for all $r > 1$ (see Figure 1). Therefore

$$\lim_{r \rightarrow +\infty} \frac{r}{\tilde{d}(0, r; A_s)} = +\infty.$$

Since \mathbb{Z} is closed and discrete in \mathbb{R} , it turns out that $\mathbb{C} \setminus A_s$ is not quasiconformally equivalent to $\mathbb{C} \setminus \mathbb{Z}$ from Theorem A. Namely A_s is not tame.

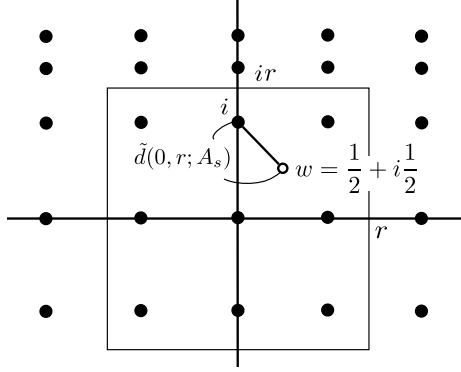


FIGURE 1. : $s \leq 1$

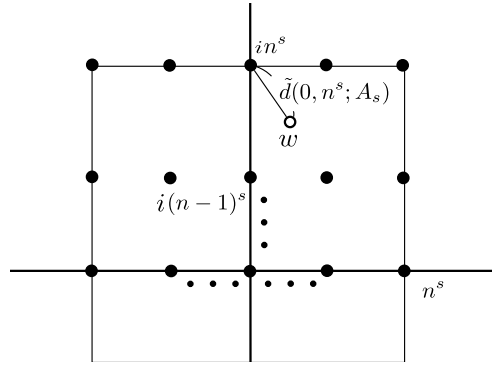


FIGURE 2. : $s > 1$

Next, if $s > 1$,

$$\left\{ 2\tilde{d}(0, n^s; A_s) \right\}^2 = (n^s - (n-1)^s)^2 + 1$$

holds for all $n \in \mathbb{N}$ (see Figure 2). Therefore

$$\begin{aligned} \frac{n^s}{\tilde{d}(0, n^s; A_s)} &= \frac{2n^s}{\sqrt{(n^s - (n-1)^s)^2 + 1}} \\ &= \frac{2}{\sqrt{\left(1 - \left(1 - \frac{1}{n}\right)^s\right)^2 + \left(\frac{1}{n}\right)^{2s}}} \rightarrow +\infty \quad (n \rightarrow +\infty). \end{aligned}$$

Similarly it is revealed that A_s is not tame. \square

Corollary 2.2. *For arbitrary $s > 0$, the discrete set $A'_s = \mathbb{Z} + i\{n^s \mid n = 0, 1, 2, \dots\}$ is not tame. In particular $\mathbb{Z} + i\mathbb{N}$ is not tame.*

Proof. Instead of $\tilde{d}(0, n^s; A_s)$ used in the proof of Corollary 2.1, we compute $\tilde{d}(n^s i/2, n^s/2; A'_s)$. It follows that

$$\begin{aligned} \frac{\frac{n^s}{2}}{\tilde{d}\left(\frac{n^s}{2}i, \frac{n^s}{2}; A'_s\right)} &= \frac{n^s}{\sqrt{(n^s - (n-1)^s)^2 + 1}} \\ &= \frac{1}{\sqrt{\left(1 - \left(1 - \frac{1}{n}\right)^s\right)^2 + \left(\frac{1}{n}\right)^{2s}}} \rightarrow +\infty \quad (n \rightarrow +\infty). \end{aligned}$$

□

2.4. Example. Let $A = \mathbb{Z} + i\{2^n \mid n = 0, 1, 2, \dots\}$. It seems that A is similar to A_s and A'_s of the above corollaries, however, we cannot decide whether A is tame or not from Theorem A.

3. PROOF OF THEOREM B

3.1. Extremal distances and Vuorinen's Theorem. Let $D \subset \hat{\mathbb{C}}$ be a domain. For given continua $E, F \subset D$,

$$\delta^D(E, F) = \text{mod}(\mathcal{F}^D(E, F))$$

is called the extremal distance between E and F in D , where mod denotes the n -modulus of a curve family and $\mathcal{F}^D(E, F)$ denotes the family of all rectifiable curves which join E and F in D . The n -modulus coincides with the reciprocal of the extremal length introduced by L. V. Ahlfors and A. Beurling [1]. It is well known that a sense preserving homeomorphism f becomes K -quasiconformal for a constant $K \geq 1$ if and only if f satisfies the following inequality for any curve family \mathcal{F} in the domain of f .

$$\frac{1}{K} \text{mod}(\mathcal{F}) \leq \text{mod}(f(\mathcal{F})) \leq K \text{mod}(\mathcal{F}).$$

The next useful lower bound for extremal distances was presented by M. Vuorinen in [9, Lemma 4.7]. For each pair of disjoint continua $E, F \subset \hat{\mathbb{C}}$, it holds that

$$\delta^{\hat{\mathbb{C}}}(E, F) \geq \frac{2}{\pi} \log \left(1 + \frac{\min\{\text{diam}(E), \text{diam}(F)\}}{\text{dist}(E, F)} \right).$$

3.2. Some lemmas. To prove Theorem B, we shall prove some lemmas.

Lemma 3.1. *Let $A \in \mathcal{P}$ and $h \in \text{Aut}(\mathbb{C} \setminus A)$. If $\text{ord}(h) = \infty$, then h can be written as $h(z) = z + b$ with a certain $b \in \mathbb{C} \setminus \{0\}$.*

Proof. From Riemann's removable singularity theorem, we can regard h as an element of $\text{Aut}(\mathbb{C})$. Therefore h can be written as $h(z) = az + b$ with certain $a \in \mathbb{C} \setminus \{0\}$ and $b \in \mathbb{C}$. It is easily seen $a = 1$ and $b \neq 0$ from the conditions $\text{ord}(h) = \infty$ and $A \in \mathcal{P}$. □

Lemma 3.2. *Let $A \in \mathcal{P}$. Assume that $z + 1 \in \text{Aut}(\mathbb{C} \setminus A)$ and $(\mathbb{C} \setminus A)/\langle z + 1 \rangle$ has infinitely many punctures. If there exists a quasiconformal homeomorphism $f : \mathbb{C} \setminus \mathbb{Z} \rightarrow \mathbb{C} \setminus A$, then f satisfies the following condition.*

$$\sup_{a \in A} |f^{-1}(a) - f^{-1}(a - 1)| = +\infty.$$

Here, we use the same letter f for the quasiconformal extension of f which maps \mathbb{C} onto \mathbb{C} (see the proof of Theorem A).

Remark that since $z + 1$ belongs to $\text{Aut}(\mathbb{C} \setminus A)$, $a - 1 \in A$ for all $a \in A$. Further, since $f(\mathbb{Z}) = A$, $f^{-1}(a) \in \mathbb{Z}$ for all $a \in A$.

Proof. To obtain a contradiction, assume that there exists a constant $M \in \mathbb{N}$ such that $|f^{-1}(a) - f^{-1}(a - 1)| \leq M$ for all $a \in A$.

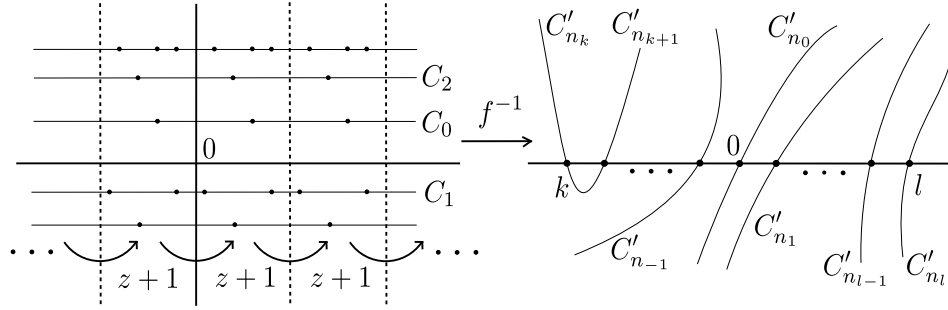
Let $S = \{x + iy \mid x \in [0, 1), y \in \mathbb{R}\}$. The assumption that $(\mathbb{C} \setminus A)/\langle z + 1 \rangle$ has infinitely many punctures means that $S \cap A$ consists of countably infinitely many points. Recall that A is closed and discrete in \mathbb{C} , so that, $\{\text{Im}(z) \mid z \in S \cap A\}$ also consists of countably infinitely many points. Numbering them suitably, we let

$$\{\text{Im}(z) \mid z \in S \cap A\} = \{a_n\}_{n=1}^{\infty},$$

and define curves C_n, C'_n by

$$C_n(t) = t + ia_n \quad (t \in \mathbb{R}), \quad C'_n = f^{-1}(C_n).$$

By definition, each curve C'_n passes integers. Conversely, for each $m \in \mathbb{Z}$, there uniquely exists a curve in $\{C'_n\}_{n=1}^{\infty}$ which passes m . We denote such a curve by C'_{n_m} .



From the properties of the curve family $\{C'_n\}_{n=1}^{\infty}$, there exist $k, l \in \mathbb{Z}$ ($k < l$) such that $\{C'_{n_k}, C'_{n_{k+1}}, \dots, C'_{n_{l-1}}, C'_{n_l}\}$ consists of $M+1$ curves. Since $|f^{-1}(a) - f^{-1}(a - 1)| \leq M$ for all $a \in A$, each curve C'_{n_i} ($i = k, k+1, \dots, l$) passes at least two points of $\{k - M, k - M + 1, \dots, k - 1\} \cup \{l + 1, l + 2, \dots, l + M\}$. There must exist at least $2(M+1)$ points to pass, however, $\{k - M, k - M + 1, \dots, k - 1\} \cup \{l + 1, l + 2, \dots, l + M\}$ consists of only $2M$ points. This is a contradiction. \square

Lemma 3.3. *Let $A \in \mathcal{P}$ be tame and $f : \mathbb{C} \setminus \mathbb{Z} \rightarrow \mathbb{C} \setminus A$ be a K -quasiconformal homeomorphism. For arbitrary $n, m \in \mathbb{Z}$ ($m > n$) and $d \in \mathbb{N}$, we set $N_d = [n - d, n]$ and $M_d = [m, m + d]$. Then*

$$\frac{\min \{\text{diam} f(N_d), \text{diam} f(M_d)\}}{|f(m) - f(n)|} \leq \exp \left(\frac{\pi^2 K}{\log \left(1 + 2 \frac{m - n}{d} \right)} \right) - 1.$$

Here, we use the same letter f for the quasiconformal extension of f which maps \mathbb{C} onto \mathbb{C} (see the proof of Theorem A).

Proof. Because $\text{dist}(f(N_d), f(M_d)) \leq |f(m) - f(n)|$, it follows from Vuorinen's Theorem that

$$\frac{2}{\pi} \log \left(1 + \frac{\min \{\text{diam} f(N_d), \text{diam} f(M_d)\}}{|f(m) - f(n)|} \right) \leq \delta^{\hat{\mathbb{C}}}(f(N_d), f(M_d)).$$

The quasiconformality of f implies that

$$\delta^{\hat{\mathbb{C}}}(f(N_d), f(M_d)) \leq K \delta^{\hat{\mathbb{C}}}(N_d, M_d).$$

Further, since N_d, M_d are separated by the ring domain $\{d/2 < |z - (n - d/2)| < d/2 + m - n\}$,

$$\delta^{\hat{\mathbb{C}}}(N_d, M_d) \leq \frac{2\pi}{\log \frac{d/2 + m - n}{d/2}} = \frac{2\pi}{\log \left(1 + 2 \frac{m - n}{d} \right)}.$$

Consequently, we obtain

$$\frac{2}{\pi} \log \left(1 + \frac{\min \{\text{diam} f(N_d), \text{diam} f(M_d)\}}{|f(m) - f(n)|} \right) \leq \frac{2\pi K}{\log \left(1 + 2 \frac{m - n}{d} \right)}.$$

Simplifying this inequality, we obtain the required inequality. \square

3.3. Proof of Theorem B. Theorem B is proved as a corollary of the next theorem.

Theorem 3.4. *Let $A \in \mathcal{P}$. Assume there exists an automorphism of infinite order $h \in \text{Aut}(\mathbb{C} \setminus A)$ such that the quotient space $(\mathbb{C} \setminus A) / \langle h \rangle$ has infinitely many punctures. If there exists a quasiconformal homeomorphism $f : \mathbb{C} \setminus \mathbb{Z} \rightarrow \mathbb{C} \setminus A$, then for arbitrary $d \in \mathbb{N}$ and $\varepsilon > 0$, there exists $n \in \mathbb{Z}$ such that*

$$\text{diam} f([n - d, n]) \leq \varepsilon.$$

Here, we use the same letter f for the quasiconformal extension of f which maps \mathbb{C} onto \mathbb{C} (see the proof of Theorem A).

Proof. By Lemma 3.1, h can be written as $h(z) = z + b$ with a certain $b \in \mathbb{C} \setminus \{0\}$. Then there exists an affine transformation $g \in \text{Aut}(\mathbb{C})$ satisfying $g \circ h \circ g^{-1}(z) = z + 1 \in \text{Aut}(\mathbb{C} \setminus g(A))$. Therefore we may assume $h(z) = z + 1$.

Let $d \in \mathbb{N}$, $\varepsilon > 0$ and $f : \mathbb{C} \setminus \mathbb{Z} \rightarrow \mathbb{C} \setminus A$ be a K -quasiconformal homeomorphism. Choose $\varepsilon' > 0$ with

$$\exp \left(\frac{\pi^2 K}{\log (1 + 2/d\varepsilon')} \right) - 1 < \varepsilon.$$

By Lemma 3.2, there exists $a \in A$ such that

$$|f^{-1}(a) - f^{-1}(a-1)| > \frac{1}{\varepsilon'}.$$

Let $n = \min\{f^{-1}(a), f^{-1}(a-1)\}$, $m = \max\{f^{-1}(a), f^{-1}(a-1)\}$. Since $|f(m) - f(n)| = 1$, it follows from Lemma 3.3 that

$$\begin{aligned} \min\{\text{diam}f(N_d), \text{diam}f(M_d)\} &\leq \exp\left(\frac{\pi^2 K}{\log\left(1 + 2\frac{m-n}{d}\right)}\right) - 1 \\ &\leq \exp\left(\frac{\pi^2 K}{\log(1 + 2/d\varepsilon')}\right) - 1 < \varepsilon, \end{aligned}$$

where $N_d = [n-d, n]$ and $M_d = [m, m+d]$.

Therefore if $\min\{\text{diam}f(N_d), \text{diam}f(M_d)\} = \text{diam}f(N_d)$, then n is the desired integer. In the other case, $m+d$ is the desired integer. \square

Theorem B. *Let $A \in \mathcal{P}$. Assume there exists an automorphism of infinite order $h \in \text{Aut}(\mathbb{C} \setminus A)$ such that the quotient space $(\mathbb{C} \setminus A)/\langle h \rangle$ has infinitely many punctures. If A is tame, then for any $\varepsilon > 0$ and $d \in \mathbb{N}$, there exists $a \in A$ such that*

$$\#D(a, \varepsilon) \cap A \geq d,$$

where for a finite set X , $\#X$ denotes the number of elements of X .

Proof. In the proof of Theorem 3.4, remark that the restriction $f|_{\mathbb{Z}} : \mathbb{Z} \rightarrow A$ is bijective. We immediately obtain the claim by taking $a = f(n)$ for n obtained in Theorem 3.4. \square

3.4. Example. It directly follows from Theorem B that the closed discrete subset A of Example 2.4 is not tame.

3.5. Example. Let

$$A = \mathbb{Z} + i \bigcup_{n=1}^{\infty} \left\{ 2^n + \left(\frac{1}{2}\right)^{n+1} e^{\frac{2\pi k i}{n}} \mid k = 0, 1, \dots, n-1 \right\}.$$

Then A satisfies the conditions that $z+1 \in \text{Aut}(\mathbb{C} \setminus A)$ and $(\mathbb{C} \setminus A)/\langle z+1 \rangle$ has infinitely many punctures, however, it cannot be decided whether A is tame or not from Theorem A and Theorem B.

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